PMM U.S.S.R.,Vol.51,No.4,pp.523-535,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

CONTACT PROBLEMS OF THE INTERACTION BETWEEN VISCOELASTIC FOUNDATIONS SUBJECTED TO AGEING AND SYSTEMS OF STAMPS NOT APPLIED SIMULTANEOUSLY*

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Plane contact problems of the interaction between inhomogeneous ageing viscoelastic foundations and arbitrary finite systems of rigid stamps, not applied and removed simultaneously, are investigated. Formulations of the problems are given. Systems of resolving two-dimensional integral equations are derived and methods are proposed for their solution. Numerical computations are presented for different kinds of ageing during interaction between a concrete foundation and two dissimilar stamps not applied simultaneously. Qualitative effects are discussed.

1. We consider the problem of the action of an arbitrary system of rigid stamps on a foundation possessing the properties of inhomogeneity, creep, and ageing /1-3/ under plane strain conditions. The times of application and removal of the system stamps are distinct. The foundations consist of two layers. The lower layer of thickness H makes contact without friction of adhesion with the rigid base, while the upper layer lies on the lower one without friction. Each i-th stamp makes contact with the section $a_i \leqslant x \leqslant b_i$ of the upper layer of thickness h, where x is the horizontal coordinate. It is assumed that all the stamps are smooth and the upper layer is thin, i.e., $b_i - a_i \gg h$ /4/. The time of application of the i-th stamp is denoted by τ_i and the instant of removal by τ_i° ($\tau_i < \tau_i^\circ$; $i = 1, 2, \ldots, n$), the force and moment acting on it are $P_i(t)$ and $M_i(t)$, respectively, where t is the time.

We will write the equations of state of a linear inhomogeneous ageing viscoelastic body in the most general form /3/

$$\begin{aligned}
&e_{ij}(\mathbf{x}, t) = \frac{s_{ij}(\mathbf{x}, t)}{2G(t+\varkappa(\mathbf{x}), \mathbf{x})} - \int_{\tau_1}^{t} \frac{s_{ij}(\mathbf{x}, \tau)}{2G(\tau+\varkappa(\mathbf{x}), \mathbf{x})} \times \\
&\quad Q_1(t+\varkappa(\mathbf{x}), \tau+\varkappa(\mathbf{x}), \mathbf{x})\, d\tau \\
&\varepsilon_{kk}(\mathbf{x}, t) = \frac{\sigma_{kk}(\mathbf{x}, t)}{E^*(t+\varkappa(\mathbf{x}), \mathbf{x})} - \int_{\tau_1}^{t} \frac{\sigma_{kk}(\mathbf{x}, \tau)}{E^*(\tau+\varkappa(\mathbf{x}), \mathbf{x})} \times \\
&\quad Q_2(t+\varkappa(\mathbf{x}), \tau+\varkappa(\mathbf{x}), \mathbf{x})\, d\tau \\
&Q_1(t, \tau) = G(\tau)\frac{\partial}{\partial\tau}\left[\frac{1}{G(\tau)} + \omega(t, \tau)\right] \\
&Q_2(t, \tau) = E^*(\tau)\frac{\partial}{\partial\tau}\left[\frac{1}{E^*(\tau)} + C^*(t, \tau)\right] \\
&E^*(t) = E(t)[1-2\nu_1(t)]^{-1}, \quad C^*(t, \tau) = [1-2\nu_2(t, \tau)]\, C(t, \tau) \\
&K(t, \tau) = E(\tau)\frac{\partial}{\partial\tau}\left[\frac{1}{E(\tau)} + C(t, \tau)\right]
\end{aligned} \tag{1.1}$$

Here $e_{ij}(\mathbf{x}, t), s_{ij}(\mathbf{x}, t)$ and $\varepsilon_{kk}(\mathbf{x}, t), \sigma_{kk}(\mathbf{x}, t)$ are the deviators and global parts of the strain and stress tensors, $Q_1(t, \tau)$, $Q_2(t, \tau)$, $\omega(t, \tau)$, $C^*(t, \tau)$ and $G(t)$ and $E^*(t)$ are the creep kernels, the creep measure, and the instantaneous elastic strain modulus under pure shear and multilateral compression, $\nu_1(t)$ and $\nu_2(t, \tau)$ are Poisson's ratios for the instantaneous elastic strain and the creep strain, $C(t, \tau)$ and $E(t)$ are the creep measure and the instantaneous elastic strain modulus under tension, $\varkappa(\mathbf{x})$ is the inhomogeneous ageing function, $\mathbf{x}$ is the radius-vector of a body point, and τ_1 is the time the load is applied.

We will assume that the upper layer is inhomogeneous and ages with depth, i.e., in the elastic and rheological characteristics $\mathbf{x} \equiv y$, where y is the vertical coordinate. The lower layer ages homogeneously and is characterized by the time of its fabrication τ_0.

It is well-known that most viscoelastic materials show almost elastic behaviour under multilateral compression. In this case we should set $Q_2 \equiv 0$ and $E^* \equiv \mathrm{const}$ in (1.1). Then on the basis of /4, 5/ and the Volterra principle /6/ for layers satisfying these properties, we obtain a system of integral equations for n stamps

*Prikl.Matem.Mekhan.,51,4,670-685,1987

$$\int_0^h \frac{1-[\nu_1^*(t+\varkappa(y),\tau+\varkappa(y),y)]^2}{E_1^*(t+\varkappa(y),\ \tau+\varkappa(y),\ y)}\,dy q_i(x,t)+ \tag{1.2}$$

$$2\,\frac{1-[\nu_2^*(t-\tau_0,\ \tau-\tau_0)]^2}{\pi E_2^*(t-\tau_0,\ \tau-\tau_0)}\sum_{j=1}^{n}\int_{a_j}^{b_j} q_j(\xi,t)\,k\left(\frac{x-\xi}{H}\right)d\xi=$$

$$\delta_i(t)+\alpha_i(t)\left(x-\frac{a_i+b_i}{2}\right)-g_i\left(x-\frac{a_i+b_i}{2}\right)$$

$$(a_i\leqslant x\leqslant b_i,\ i=1,\ldots,n)$$

The additional conditions take the form

$$\int_{a_i}^{b_i} q_i(x,t)\,dx=P_i(t),\quad \int_{a_i}^{b_i}\left(x-\frac{a_i+b_i}{2}\right)q_i(x,t)\,dx=M_i(t) \tag{1.3}$$

Here $q_i(x,t)\equiv 0$ in (1.2) and (1.3) when $\tau_i^\circ<t<\tau_i$.

Here ν_k^* and E_k^* are integral operators describing the material of the upper layer $(k=1)$ and lower layer $(k=2)$, $q_i(x,t)$ are contact stresses under the i-th stamp, while $\delta_i(t)$, $\alpha_i(t)$, $g_i(x-(a_i+b_i)/2)$ are the settling, rotation, and shape of the base of this stamp, and $k((x-\xi)/H)$ is the kernel of the contact problem under the condition that the homogeneous layer lies without friction on the rigid base /5/. The form of ν_k^* and E_k^* $(k=1,2)$ and methods of decoding and inverting the expressions containing them are known /7/; consequently, we merely note that all the operations in time are referred to the Volterra operators.

A number of materials, including concrete, are described quite well by Eqs.(1.1) under the conditions $\nu_1(t)=\nu_2(t,\tau)=\nu=\text{const}$ /8, 9/. Then for a base consisting a layers with Poisson's ratios independent of time, where ν_k and $E_k(t)$ are Poisson's ratios and instantaneous elastic strain moduli under tension of the upper $(k=1)$ and lower $(k=2)$ layers, we will have /4, 5, 8/

$$(1-\nu_1^2)\int_0^h\left[\frac{q_i(x,t)}{E_1(t+\varkappa(y),y)}-\int_{\tau_i}^{t}\frac{q_i(x,\tau)}{E_1(\tau+\varkappa(y),y)}\times K_1(t+\varkappa(y),\tau+\varkappa(y),y)\,d\tau\right]dy+ \tag{1.4}$$

$$2\,\frac{1-\nu_2^2}{\pi}\sum_{j=1}^{n}\left[\int_{a_j}^{b_j}\frac{q_j(\xi,t)}{E_2(t-\tau_0)}\,k\left(\frac{x-\xi}{H}\right)d\xi-\int_{\tau_i}^{t}\int_{a_j}^{b_j}\frac{q_j(\xi,\tau)}{E_2(\tau-\tau_0)}\,k\left(\frac{x-\xi}{H}\right)d\xi K_2(t-\tau_0,\tau-\tau_0)\,d\tau\right]=$$

$$\delta_i(t)+\alpha_i(t)\left(x-\frac{a_i+b_i}{2}\right)-g_i\left(x-\frac{a_i+b_i}{2}\right)$$

$$q_i(x,t)\equiv 0,\ \tau_i^\circ<t<\tau_i\ \ (a_i\leqslant x\leqslant b_i,\ i=1,\ldots,n)$$

The additional conditions retain the form of (1.3). Bases of two kinds can still be considered when the upper layer shows almost elastic behaviour while the lower layer has a constant Poisson's ratio, and vice versa. The systems of contact problem integral equations for such bases can be obtained by interchanging the places of the terms outside the integral with respect to the coordinate in (1.2) and (1.4). It should be recalled that a smooth contact is always assumed on its lower face for a homogeneous layer when $Q_2\equiv 0$, $E^*\equiv\text{const}$, because only then is the kernel $k((x-\xi)/H)$ independent of the time and its form is known /5/. If $\nu_2=\text{const}$ for the lower homogeneous layer then it can also be linked with a non-deformable foundation. The contact problem kernel here is independent of time as before.

It should be noted that problems regarding rough viscoelastic foundations, foundations containing a pin-joint layer /10, 11/, and contact problems of the wear of elastic and viscoelastic foundations by a system of stamps not applied simultaneously result in analogous systems of equations.

The systems of equations of the above problems can be written in a single form containing Volterra operators in time and completely continuous, selfadjoint, and positive-definite operators in the coordinate.

To be specific, we will consider the system of Eqs.(1.4) with the additional conditions (1.3) and we will reduce them to general form. We make the change of variables

$$x^*=\frac{2x-a_i-b_i}{b_i-a_i},\quad \xi^*=\frac{2\xi-a_i-b_i}{b_i-a_i}\quad (a_i\leqslant x,\xi\leqslant b_i) \tag{1.5}$$

$$\frac{2H}{b_1-a_1}=\lambda,\quad \frac{a_j+b_j}{b_1-a_1}=\eta_j,\quad \frac{b_j-a_j}{b_1-a_1}=\zeta_j$$

$$k_{ij}(x^*, \xi^*) = \pi^{-1}k\left(\frac{\zeta_i x^* + \eta_i - \zeta_j\xi^* - \eta_j}{\lambda}\right) = \pi^{-1}k\left(\frac{x-\xi}{H}\right)$$

$$q_i^*(x^*, t^*) = \frac{2q_i(x,t)(1-\nu_2^2)}{E_2(t-\tau_0)}, \quad g_i^*(x^*) = \frac{2g_i(x-(a_i+b_i)/2)}{b_1-a_1}$$

$$\delta_i^*(t^*) = \frac{2\delta_i(t)}{b_1-a_1}, \quad \zeta_i x^* = \frac{2x-a_i-b_i}{b_1-a_1}$$

$$t^* = t\tau_1^{-1}, \quad \tau^* = \tau\tau_1^{-1}, \quad \tau_i^* = \tau_i\tau_1^{-1}, \quad \tau_0^* = \tau_0\tau_1^{-1}, \quad (\tau_i^\circ)^* = \tau_i^\circ\tau_1^{-1}$$

$$c^*(t^*) = \frac{(1-\nu_1^2)E_2(t-\tau_0)h}{(1-\nu_2^2)E_1^\circ(t)(b_1-a_1)}$$

$$K_1^*(t^*, \tau^*) = \frac{E_1^\circ(t)E_2(\tau-\tau_0)}{E_1^\circ(\tau)E_2(t-\tau_0)}K_1^\circ(t,\tau)\tau_1$$

$$M_i^*(t^*) = \frac{8M_i(t)(1-\nu_2^2)}{E_2(t-\tau_0)(b_i-a_i)^2}, \quad P_i^*(t^*) = \frac{4P_i(t)(1-\nu_2^2)}{E_2(t-\tau_0)(b_i-a_i)}$$

$$\frac{1}{E_1^\circ(t)} = \frac{1}{h}\int_0^h \frac{dy}{E_1(t+\varkappa(y), y)}$$

$$K_1^\circ(t,\tau) = \frac{E_1^\circ(\tau)}{h}\int_0^h \frac{K_1(t+\varkappa(y), \tau+\varkappa(y), y)}{E_1(\tau+\varkappa(y), y)}dy$$

$$K_2^*(t^*, \tau^*) = K_2(t-\tau_0, \tau-\tau_0)\tau_1, \quad -1 \leqslant x^*, \xi^* \leqslant 1$$

omitting the asterisks in (1.5) and setting

$$\mathbf{L}_k^*(\tau_i, t)w(t) = \int_{\tau_i}^t w(\tau)K_k(t,\tau)\,d\tau$$

$$\mathbf{A}_{ij}^*v(x) = \int_{-1}^1 k_{ij}(x,\xi)v(\xi)\,d\xi \quad (i, j = 1, \ldots, n; \quad k = 1, 2)$$

we obtain ($\mathbf{I}^*$ is the identity operator)

$$c(t)(\mathbf{I}^* - \mathbf{L}_1^*(\tau_i, t))q_i(x,t) + (\mathbf{I}^* - \mathbf{L}_2^*(1, t))\sum_{j=1}^n \mathbf{A}_{ij}^*q_j(x,t) = \delta_i(t) + \alpha_i(t)\zeta_i x - g_i(x) \tag{1.6}$$

$$q_i(x,t) \equiv 0, \quad \tau_i^\circ < t < \tau_i \quad (i = 1, \ldots, n)$$

$$\int_{-1}^1 q_i(x,t)\,dx = P_i(t), \quad \int_{-1}^1 q_i(x,t)\,x\,dx = M_i(t) \tag{1.7}$$

Relationships (1.6) and (1.7) yield a step-by-step process for obtaining the system of resolving equations of the problem, where we have k equations with k unknowns at the time of interaction of k stamps with the foundation and the whole loading history of the foundation is here taken into account. The system of equations is obtained at each step individually for the specific problem.

For instance, we will consider the successive attachment of single stamps. Then for $\tau_k \leqslant t < \tau_{k+1}$ (1.6) takes the form

$$c(t)(\mathbf{I}^* - \mathbf{L}_1^*(\tau_k, t))q_i(x,t) + (\mathbf{I}^* - \mathbf{L}_2^*(\tau_k, t))\sum_{j=1}^k \mathbf{A}_{ij}^*q_j(x,t) = \delta_i(t) + \alpha_i(t)\zeta_i x - g_i(x) + c(t)\mathbf{L}_1^*(\tau_i, \tau_k)q_i(x,t) + \sum_{j=1}^{k-1}\mathbf{L}_2(\tau_j, \tau_k)\mathbf{A}_{ij}^*q_j(x,t) \quad (i = 1, \ldots, k) \tag{1.8}$$

where the components on the right-hand side of (1.8) containing $q_m(x,t)$ are found from the solution of the problem in the preceding step and depend on x and t. They determine the surface distortion of the viscoelastic foundation because of material creep. In fact the solution (1.8) is equivalent to the solution of the problem of attaching k stamps with bottom shapes defined by the last three terms on the right-hand side of (1.8), to a deformable base simultaneously at the time τ_k.

Thus, to investigate the arbitrary process of attachment or removal of stamps, it is necessary to solve a system of equations of the following kind at each step:

$$c(t)(\mathbf{I}^* - \mathbf{L}_1^*)q^i(x,t) + (\mathbf{I}^* - \mathbf{L}_2^*)\sum_{j=1}^n \mathbf{A}_{ij}^*q^j(x,t) = \delta^i(t) + \alpha^i(t)x - g^i(x,t), \quad \mathbf{L}_m^* = \mathbf{L}_m^*(\tau_k, t) \tag{1.9}$$

$$(i = 1, \ldots, n; \; m = 1, 2)$$

Conditions (1.7) with the superscript i are the additional conditions for (1.9).

Later we will examine the solution of the system of Eqs. (1.9). We will denote the operators, tensors, and vectors by capital letters with an asterisk, by capital letters and by small letters, respectively, in heavy type. We shall also use tensor analysis symbolism from /12/.

2. We consider the vector-function $\mathbf{a}(x) = a^i(x)\mathbf{i}^i$ and the tensor-function of two variables $\mathbf{K}(x, \xi) = k^{ij}(x, \xi)\mathbf{i}^i\mathbf{i}^j$, where $\mathbf{i}^k$ is the orthonormalized algebraic vector basis of the N-dimensional Euclidean space V. Here and henceforth the summation will be carried out over repeated superscripts, and the superscripts themselves take natural values from 1 to N.

We will introduce a complete Hilbert space $L_2([-1, 1], V)$ of vector-functions with the following global scalar product and norm /13, 14/:

$$\mathbf{a}(x), \mathbf{b}(x) \in L_2([-1, 1], V) : (\mathbf{a}(x), \mathbf{b}(x)) = \int_{-1}^{1} \mathbf{a}(x)\cdot\mathbf{b}(x)\,dx$$
$$\|\mathbf{a}(x)\| = (\mathbf{a}(x), \mathbf{a}(x))^{1/2} < \infty, \quad \mathbf{a}(x)\cdot\mathbf{b}(x) = a^k(x)\,b^k(x)$$

We similarly introduce the complete Hilbert space of tensor-functions of two variables

$$\mathbf{K}(x, \xi),\ \mathbf{M}(x, \xi) \in \mathbf{L}_2([-1, 1], V) : [\mathbf{K}(x, \xi), \mathbf{M}(x, \xi)] =$$
$$\int_{-1}^{1}\int_{-1}^{1} \mathbf{K}(x, \xi)\cdot\cdot\mathbf{M}(x, \xi)\,d\xi\,dx, \quad |\mathbf{K}(x, \xi)| = [\mathbf{K}(x, \xi), \mathbf{K}(x, \xi)]^{1/2}$$
$$\mathbf{K}(x, \xi)\cdot\cdot\mathbf{K}(x, \xi) = k^{ij}(x, \xi)\,k^{ij}(x, \xi)$$

We recall that /13/

$$\int_a^b \mathbf{a}(x)\,dx = \int_a^b a^k(x)\,dx\,\mathbf{i}^k, \quad \frac{\partial^n}{\partial x^n}\mathbf{a}(x) = \frac{\partial^n a^k(x)}{\partial x^n}\mathbf{i}^k, \quad \mathbf{a}(x)f(x) = [a^k(x)f(x)]\,\mathbf{i}^k$$

and the vector-function is continuous in x if and only if all its components are continuous in x.

We designate the expression

$$(\mathbf{K}(x, \xi), \mathbf{a}(\xi)) = \int_{-1}^{1} \mathbf{K}(x, \xi)\cdot\mathbf{a}(\xi)\,d\xi \tag{2.1}$$

the global scalar product (on the right) of a tensor-function by a vector-function.

Lemma **1°**. The vector-function $\mathbf{a}(x) \in L_2([-1, 1], V)$ if and only if its components $a^k(x) \in L_2[-1, 1]$, where $L_2[-1, 1]$ is a space of functions square integrable in the segment $[-1, 1]$ /13/.

2°. The tensor-function $\mathbf{K}(x, \xi) \in \mathbf{L}_2([-1, 1], V)$ if and only if its components $k^{ij}(x, \xi) \in \mathbf{L}_2[-1, 1]$, where $\mathbf{L}_2[-1, 1]$ is a space of functions integrable with its second degree in the square $[-1, \leqslant x \leqslant 1, -1 \leqslant \xi \leqslant 1]$.

The following theorem can be proved on the basis of the lemmas.

Theorem 1. The operator $\mathbf{A}^*$ formed by the global scalar product (2.1), i.e.,

$$\mathbf{A}^*\mathbf{a}(x) = (\mathbf{K}(x, \xi), \mathbf{a}(\xi)), \quad \mathbf{K}(x, \xi) \in \mathbf{L}_2([-1, 1], V)$$

is completely continuous from $L_2([-1, 1], V)$ into $L_2([-1, 1], V)$; if $\mathbf{K}(x, \xi) = \mathbf{K}^T(\xi, x)$ the operator $\mathbf{A}^*$ is selfadjoint.

We will examine the question of the expansion of functions from $L_2([-1, 1], V)$ and $\mathbf{L}_2([-1, 1], V)$ in series in functional vector bases. Let $\{P_k^*(x)\}$ be a basis of $L_2[-1, 1]$. Then (see the lemma)

$$\mathbf{a}(x) = a^i(x)\,\mathbf{i}^i = \sum_{k=0}^{\infty} a_k{}^i P_k{}^*(x)\,\mathbf{i}^i = \sum_{k=0}^{\infty} a_k{}^i \mathbf{p}_k{}^i(x) \tag{2.2}$$
$$\mathbf{p}_k{}^i(x) = P_k{}^*(x)\,\mathbf{i}^i, \quad (\mathbf{p}_k{}^i(x), \mathbf{p}_n{}^j(x)) = \begin{cases} 1; & k = n, i = j \\ 0; & \text{otherwise} \end{cases}$$

where $\mathbf{p}_k{}^i(x)$ $(k = 0, \ldots, \infty)$ is a basis of $L_2([-1, 1], V)$.

Similarly,

$$\mathbf{K}(x, \xi) = k^{ij}(x, \xi)\,\mathbf{i}^i\mathbf{i}^j = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn}^{ij} P_m{}^*(x)\,P_n{}^*(\xi)\,\mathbf{i}^i\mathbf{i}^j = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn}^{ij}\mathbf{p}_m{}^i(x)\,\mathbf{p}_n{}^j(\xi) \tag{2.3}$$

where if $\mathbf{K}(x, \xi) = \mathbf{K}^T(\xi, x)$, then $r_{mn}^{ij} = r_{nm}^{ji}$.

We consider the following equation

$$c(t)(\mathbf{I}^* - \mathbf{L}_1^*)\mathbf{q}(x, t) + (\mathbf{I}^* - \mathbf{L}_2^*)\mathbf{A}^*\mathbf{q}(x, t) = \boldsymbol{\delta}(t) + \boldsymbol{\alpha}(t)x - \mathbf{g}(x, t) \tag{2.4}$$

with the additional conditions

$$\int_{-1}^{1} \mathbf{q}(x, t)\,dx = \mathbf{p}(t), \quad \int_{-1}^{1} \mathbf{q}(x, t)\,x\,dx = \mathbf{m}(t) \tag{2.5}$$

where $c(t) > 0$ is a function continuous in t, $\mathbf{q}(x, t)$, $\mathbf{g}(x, t)$ are vector-functions continuous in t in $L_2([-1, 1], V)$, $\mathbf{p}(t)$, $\mathbf{m}(t)$, $\boldsymbol{\delta}(t)$ and $\boldsymbol{\alpha}(t)$ are vector-functions continuous in t with values from V, $\mathbf{A}^*$ is a selfadjoint, completely continuous and positive-definite operator from $L_2([-1, 1], V)$ into $L_2([-1, 1], V)$, $t \in [\tau_r, \tau_{r+1}]$, the kernels of the Volterra operators $\mathbf{L}_1^*$ and $\mathbf{L}_2^*$ are continuous in the large or have a weak singularity, where $K_k(t, \tau) = A_k(t, \tau)(t-\tau)^{-m_k}$, where $A_k(t, \tau)$ are continuous functions, $0 \leqslant m_k < 1$ $(k = 1, 2)$ /15/.

Let only $\mathbf{q}(x, t)$, $\mathbf{p}(t)$ and $\mathbf{m}(t)$ be unknown in (2.4) and (2.5). Furthermore, we consider the subscripts (if this is not specially stipulated) to vary between 0 and ∞ while the sign Σ denotes summation over one of the repeated subscripts when it runs through all its values. Utilizing the classical method of the thoery of operators in Hilbert spaces /16, 17/, we represent the solution in the form (see /18/ also)

$$\begin{aligned} &\mathbf{q}(x, t) = \Sigma\omega_i(t)\,\boldsymbol{\varphi}_i(x), \quad \mathbf{g}(x, t) = \Sigma g_i^\circ(t)\,\boldsymbol{\varphi}_i(x) \\ &\boldsymbol{\delta}(t) = \delta^k(t)\,\mathbf{i}^k = \delta^k(t)\,\Sigma\delta_i^k\boldsymbol{\varphi}_i(x) \\ &\boldsymbol{\alpha}(t)\,x = \alpha^k(t)\,x\mathbf{i}^k = \alpha^k(t)\,\Sigma X_i^k\boldsymbol{\varphi}_i(x) \end{aligned} \tag{2.6}$$

where $\boldsymbol{\varphi}_i(x)$ are the eigenvector-functions of the operator $\mathbf{A}^*$ corresponding to its eigennumbers $\alpha_i^\circ > 0$, i.e.

$$\mathbf{A}^*\boldsymbol{\varphi}_i(x) = \alpha_i^\circ\boldsymbol{\varphi}_i(x) \tag{2.7}$$

It is well-known that $\{\boldsymbol{\varphi}_i(x)\}$ comprises a basis in $L_2([-1, 1], V)$. Taking (2.7) into account while substituting (2.6) into (2.4), we obtain

$$\begin{aligned} &\omega_i(t) = (\mathbf{I}^* + \mathbf{N}_i^*)\,\Omega_i(t) \\ &\Omega_i(t) = [\delta^k(t)\,\delta_i^k + \alpha^k(t)\,X_i^k - g_i^\circ(t)]/[\alpha_i^\circ + c(t)] \\ &\mathbf{N}_k^* f(t) = \int_{\tau_r}^{t} f(\tau)\,R_k^\circ(t, \tau)\,d\tau, \quad t \in [\tau_r, \tau_{r+1}] \end{aligned} \tag{2.8}$$

where $R_k^\circ(t, \tau)$ is the resolvent of the kernel

$$K_k^\circ(t, \tau, \alpha_k^\circ) = [c(t)\,K_1(t, \tau) + \alpha_k^\circ\,K_2(t, \tau)]/[c(t) + \alpha_k^\circ]$$

and the remaining quantities and determined from (2.5) and (2.6).

We now construct the eigenvector-functions and eigennumbers of the operator $\mathbf{A}^*$. We take $\boldsymbol{\varphi}_p(x)$ in the form (see (2.2))

$$\boldsymbol{\varphi}_p(x) = \Sigma a_{k(p)}^{l}\mathbf{p}_k^{l}(x) \tag{2.9}$$

and $\mathbf{K}(x, \xi)$ in the form (2.3) and we substitute into (2.7), we then arrive at an algebraic system of equations with the symmetric matrices (see (2.3))

$$\Sigma r_{mn}^{ij}a_{n(p)}^{j} = \alpha_p^\circ a_{m(p)}^{i} \tag{2.10}$$

to find α_p° and the coefficients of the eigenvector-functions expansions in the functional vector basis. Limiting ourselves to k terms of the basis we obtain the k-th approximation of the Bubnov-Galerkin method /19/.

Taking the system of Legendre polynomials $P_m^*(x)$ as basis of $L_2[-1, 1]$, we obtain, written component by component

$$\begin{aligned} &q^k(x, t) = \Sigma\omega_i(t)\,\varphi_i^k(x) = \Sigma\omega_i(t)\,\Sigma a_{j(i)}^{k}P_j^*(x) \\ &p^k(t) = \Sigma\omega_i(t)\,\delta_i^k = 2^{1/2}\Sigma\omega_i(t)\,a_{0(i)}^{k} \\ &m^k(t) = \Sigma\omega_i(t)\,X_i^k = (2/3)^{1/2}\Sigma\omega_i(t)\,a_{1(i)}^{k} \end{aligned}$$

where $q^k(x, t)$ are functions continuous in t in $L_2[-1, 1]$ and $p^k(t)$ and $m^k(t)$ are continuous in t.

3. Now let $\mathbf{q}(x, t)$, $\boldsymbol{\delta}(t)$ and $\boldsymbol{\alpha}(t)$ be unknown in (2.4) and (2.5). We represent $L_2([-1, 1], V)$

in the form of a sum of orthogonal subspaces, i.e.,

$$L_2([-1, 1], V) = H(1, V) \oplus H(x, V) \oplus L_2^\circ([-1, 1], V)$$

where $H(1, V)$ is the space of algebraic vectors, $H(x, V)$ is the space of vector-functions formed by multiplying the algebraic vectors by x ($x \in [-1, 1]$), and $L_2^\circ([-1, 1], V)$ is a complete Hilbert space of vector-functions from $L_2([-1, 1], V)$, and orthogonal vector-functions from $H(1, V)$ and $H(x, V)$ /14/.

We note that $\mathbf{p}_0^k(x) = 2^{-1/2}\mathbf{i}^k$ and $\mathbf{p}_1^k(x) = (3/2)^{1/2} x\, \mathbf{i}^k$ respectively, are the functional vector bases $H(1, V)$ and $H(x, V)$, and $\{\mathbf{p}_m^k(x)\}$, say, can be taken as the basis of $L_2^\circ([-1, 1], V)$ (unless otherwise stated, the subscripts in Sect.3 vary between 2 and ∞), where $P_k^*(x)$ in (2.2) are orthonormalized Legendre polynomials.

The following theorem can be proved.

Theorem 2. 1°. The kernel of the oerator $\mathbf{A}^*$ can be represented in the form

$$\begin{aligned}\mathbf{K}(x, \xi) = \mathbf{K}_1(x, \xi) + \mathbf{k}_1^i(x)\mathbf{p}_1^i(\xi) + \mathbf{p}_1^i(x)\mathbf{k}_1^i(\xi) + \\ \mathbf{p}_0^i(x)\mathbf{k}_0^i(\xi) + \mathbf{k}_0^i(x)\mathbf{p}_0^i(\xi) + D^{ij}\mathbf{p}_0^i(x)\mathbf{p}_0^j(\xi) + \\ F^{ij}(\mathbf{p}_0^i(x)\mathbf{p}_1^j(\xi) + \mathbf{p}_1^j(x)\mathbf{p}_0^i(\xi)) + E^{ij}\mathbf{p}_1^i(x)\mathbf{p}_1^j(\xi)\end{aligned} \tag{3.1}$$

where $\mathbf{k}_m^i(x) \in L_2^\circ([-1, 1], V)$ $(m = 0, 1)$; D^{ij}, F^{ij} and E^{ij} are constants; $(\mathbf{K}_1(x, \xi), \mathbf{p}_m^i(\xi)) = 0$ $(m = 0, 1)$.

2°. The operator

$$\mathbf{B}^*: \mathbf{B}^*\mathbf{f}(x) = (\mathbf{K}_1(x, \xi), \mathbf{f}(\xi))$$

is completely continuous, selfadjoint, and positive-definite from $L_2^\circ([-1, 1], V)$ into $L_2^\circ([-1, 1], V)$; the eigenvector-functions $\boldsymbol{\psi}_k(x)$ of the operator $\mathbf{B}^*$ and its corresponding eigennumbers β_k°, i.e.,

$$\mathbf{B}^*\boldsymbol{\psi}_k(x) = \beta_k^\circ\boldsymbol{\psi}_k(x)$$

form a basis in $L_2^\circ([-1, 1], V)$.

3°. If $\mathbf{q}(x, t)$ and $\mathbf{g}(x, t)$ are continuous vector-functions in t in $L_2([-1, 1], V)$ they can be represented in the form

$$\begin{aligned}\mathbf{q}(x, t) = z_0^i(t)\mathbf{p}_0^i(x) + z_1^i(t)\mathbf{p}_1^i(x) + \Sigma z_k(t)\boldsymbol{\psi}_k(x) \\ \mathbf{g}(x, t) = g_0^i(t)\mathbf{p}_0^i(x) + g_1^i(t)\mathbf{p}_1^i(x) + \Sigma g_k(t)\boldsymbol{\psi}_k(x)\end{aligned} \tag{3.2}$$

where $z_m^i(t)$, $g_m^i(t)$ $(m = 0, 1)$, $z_k(t)$, $g_k(t)$ are continuous functions of t.

Regarding the first part of the theorem, we refer just to (2.3) and note that

$$\begin{aligned}\mathbf{K}_1(x, \xi) = \Sigma\Sigma r_{mn}^{ij}\mathbf{p}_m^i(x)\mathbf{p}_n^j(\xi), \ \mathbf{k}_l^i(x) = \Sigma r_{ln}^{ij}\mathbf{p}_n^j(x) \\ r_{00}^{ij} = D^{ij}, \quad r_{11}^{ij} = E^{ij}, \quad r_{01}^{ij} = F^{ij} \quad (l = 0, 1)\end{aligned} \tag{3.3}$$

The assertions of the second and third parts are based on (3.3), and the relations

$$(\mathbf{A}^*\mathbf{f}(x), \mathbf{g}(x)) = (\mathbf{B}^*\mathbf{f}(x), \mathbf{g}(x)), \ \forall \mathbf{f}(x), \mathbf{g}(x) \in L_2^\circ([-1, 1], V)$$

are known facts of the spectral theory of operators /16, 17, 18/.

We note that (3.3) and the remark preceding Theorem 2 enable us to construct $\boldsymbol{\psi}_k(x)$ and β_k° as in Sect.2, i.e.,

$$\boldsymbol{\psi}_k(x) = \Sigma b_{p(k)}^l\mathbf{p}_p^l(x), \quad \Sigma r_{mn}^{ij}b_{n(k)}^j = \beta_k^\circ b_{m(k)}^i$$

Taking account of Theorem 2 and the representations

$$\begin{aligned}\mathbf{k}_l^i(x) = \Sigma k_{k(l)}^i\boldsymbol{\psi}_k(x) \quad (l = 0, 1) \\ \boldsymbol{\delta}(t) = 2^{1/2}\delta^i(t)\mathbf{p}_0^i(x), \quad \boldsymbol{\alpha}(t)x = (^3/_2)^{-1/2}\alpha^i(t)\mathbf{p}_1^i(x)\end{aligned} \tag{3.4}$$

we obtain for the desired vector-functions (see (3.1)-(3.4))

$$\begin{aligned}&z_k(t) = -(\mathbf{I}^* + \mathbf{D}_k^*)\{[g_k(t) + (\mathbf{I}^* - \mathbf{L}_2^*)(z_1^i(t)k_{k(1)}^i + \\ &\quad z_0^i(t)k_{k(0)}^i)]/[c(t) + \beta_k^\circ]\} \\ &\mathbf{D}_k^*f(t) = \int_{\tau_r}^t f(\tau)R_k^1(t, \tau)\,d\tau, \quad t \in [\tau_r, \tau_{r+1}] \\ &\delta^i(t) = 2^{-1/2}[c(t)(\mathbf{I}^* - \mathbf{L}_1^*)z_0^i(t) + (\mathbf{I}^* - \mathbf{L}_2^*)(\Sigma z_k(t)k_{k(0)}^i + D^{ij}z_0^j(t) + F^{ij}z_1^j(t)) + g_0^i(t)]\end{aligned}$$

$$\alpha^i(t) = (^3/_2)^{1/2}[c(t)(\mathbf{I}^* - \mathbf{L}_1^*)z_1^i(t) + (\mathbf{I}^* - \mathbf{L}_2^*)(\Sigma z_k(t)k^i_{k(1)} + E^{ij}z_1^j(t) + F^{ji}z_0^j(t)) + g_1^i(t)]$$
$$z_0^i(t) = 2^{-1/2}p^i(t), \quad z_1^i(t) = (^3/_2)^{1/2}m^i(t)$$

where $R_k^1(t, \tau)$ is the resolvent of the kernel $K_k^\circ(t, \tau, \beta_k^\circ)$, $z_k(t)$, $\delta^i(t)$, $\alpha^i(t)$ are functions continuous in t by virtue of the continuity of $p^i(t)$, $m^i(t)$, $c(t)$ and the conditions imposed on the kernel of Volterra operators.

Thus, vector-functions $\delta(t)$ and $\alpha(t)$ continuous in t and the vector-function $\mathbf{q}(x, t)$ continuous in t in $L_2([-1, 1], V)$ have been found, where its components are continuous in t in $L_2[-1, 1]$.

4. We assume that $\delta(t)$, $\mathbf{m}(t)$ and $\mathbf{q}(x, t)$ are unknown in (2.4) and (2.5). Let

$$L_2^1([-1, 1], V) = H(x, V) \oplus L_2^\circ([-1, 1], V)$$

Here $\{\mathbf{p}_m^k(x)\}$ can be a basis of $L_2^1([-1, 1], V)$ (here and henceforth in Sect.4 the subscripts take values from 1 to ∞).

Theorem 3. 1°. The kernel of the operator $\mathbf{A}^*$ can be represented in the form

$$\mathbf{K}(x, \xi) = \mathbf{K}_2(x, \xi) + \mathbf{p}_0^i(x)\mathbf{k}^i(\xi) + \mathbf{k}^i(x)\mathbf{p}_0^i(\xi) + D^{ij}\mathbf{p}_0^i(x)\mathbf{p}_0^j(\xi)$$

where $\mathbf{k}^i(x) \in L_2^1([-1, 1], V)$; D^{ij} are constants, $(\mathbf{K}_2(x, \xi), \mathbf{p}_0^i(\xi)) = 0$,

$$\mathbf{K}_2(x, \xi) = \Sigma\Sigma r^{ij}_{mn}\mathbf{p}_m^i(x)\mathbf{p}_n^j(\xi), \quad \mathbf{k}^i(x) = \Sigma r^{ij}_{0n}\mathbf{p}_n^j(x), \quad D^{ij} = r^{ij}_{00}$$

2°. The operator

$$\mathbf{C}^*: \mathbf{C}^*\mathbf{f}(x) = (\mathbf{K}_2(x, \xi), \mathbf{f}(\xi))$$

is completely continuous, selfadjoint, and positive-definite from $L_2^1([-1, 1], V)$ into $L_2^1([-1, 1], V)$; the eigenvector-functions $\chi_k(x)$ of the operator $\mathbf{C}^*$ corresponding to its eigennumbers γ_k°, i.e.,

$$\mathbf{C}^*\chi_k(x) = \gamma_k^\circ\chi_k(x)$$

form a basis in $L_2^1([-1, 1], V)$.

3°. If $\mathbf{q}(x, t)$ and $(\alpha(t)x - \mathbf{g}(x, t))$ and vector-functions continuous in t in $L_2([-1, 1], V)$ they can be represented in the form

$$\mathbf{q}(x, t) = w_0^i(t)\mathbf{p}_0^i(x) + \Sigma w_k(t)\chi_k(x)$$
$$\alpha(t)x - \mathbf{g}(x, t) = d_0^i(t)\mathbf{p}_0^i(x) + \Sigma d_k(t)\chi_k(x)$$

where $w_0^i(t)$, $w_k(t)$, $d_0^i(t)$, and $d_k(t)$ are continuous in t.

Reasoning analogous to that in Sect.3 should be used in proving the theorem, while to construct the eigenvector-function and eigennumbers we will have

$$\chi_k(x) = \Sigma c^l_{p(k)}\mathbf{p}_p^l(x), \quad \Sigma r^{ij}_{mn}c^j_{n(k)} = \gamma_k^\circ c^i_{m(k)}$$

By virtue of the fact that

$$\mathbf{k}^i(x) = \Sigma k_k^i\chi_k(x), \quad \delta(t) = 2^{1/2}\delta^i(t)\mathbf{p}_0^i(x), \quad x\mathbf{i}^i = \Sigma X_k^i\chi_k(x)$$

and taking account of Theorem 3, we obtain from (2.4) and (2.5)

$$w_k(t) = (\mathbf{I}^* + \mathbf{V}_k^*)\{[d_k(t) - (\mathbf{I}^* - \mathbf{L}_2^*)k_k^i w_0^i(t)]/[c(t) + \gamma_k^\circ]\}$$
$$\mathbf{V}_k^*f(t) = \int_{\tau_r}^{t} f(\tau)R_k^2(t, \tau)\,d\tau, \quad t \in [\tau_r, \tau_{r+1}]$$
$$\delta^i(t) = 2^{-1/2}[c(t)(\mathbf{I}^* - \mathbf{L}_1^*)w_0^i(t) + (\mathbf{I}^* - \mathbf{L}_2^*)(\Sigma w_k(t)k_k^i + D^{ij}w_0^j(t)) - d_0^i(t)]$$
$$w_0^i(t) = 2^{-1/2}p^i(t), \quad m^i(t) = \Sigma X_k^i w_k(t) = (^2/_3)^{1/2}\Sigma w_k(t)c^i_{1(k)}$$

where $R_k^2(t, \tau)$ is the resolvent of the kernel $K_k^\circ(t, \tau, \gamma_k^\circ)$.

5. We will examine one more case when $\alpha(t)$, $\mathbf{p}(t)$ and $\mathbf{q}(x, t)$ are unknown in (2.4) and (2.5). We set

$$L_2^2([-1, 1], V) = H(1, V) \oplus L_2^\circ([-1, 1], V)$$

where $\{\mathbf{p}_m^k(x)\}$ can be taken as the basis of $L_2^2([-1, 1], V)$ (the subscripts in Sect.5 take all values from 0 to ∞ except unity).

Theorem 4. 1°. The kernel of the operator $\mathbf{A}^*$ can be represented in the form

$$\mathbf{K}(x,\xi)=\mathbf{K}_3(x,\xi)+\mathbf{p}_1{}^i(x)\,\mathbf{k}_*{}^i(\xi)+\mathbf{k}_*{}^i(x)\,\mathbf{p}_1{}^i(\xi)+E^{ij}\mathbf{p}_1{}^i(x)\,\mathbf{p}_1{}^j(\xi)$$

where $\mathbf{k}_*{}^i(x)\in L_2{}^2([-1,1],V)$: E^{ij} are constants; $(\mathbf{K}_3(x,\xi),\ \mathbf{p}_1{}^i(\xi))=0$;

$$\mathbf{K}_3(x,\xi)=\Sigma\Sigma r_{mn}^{ij}\mathbf{p}_m{}^i(x)\,\mathbf{p}_n{}^j(\xi),\quad \mathbf{k}_*{}^i(x)=\Sigma r_{1m}^{ij}\mathbf{p}_m{}^j(x),\quad E^{ij}=r_{11}^{ij}$$

2°. The operator

$$\mathbf{F}^*:\ \mathbf{F}^*\mathbf{f}(x)=(\mathbf{K}_3(x,\xi),\ \mathbf{f}(\xi))$$

is completely continuous, selfadjoint, and positive-definite from $L_2{}^2([-1,1],V)$ into $L_2{}^2([-1,1],V)$; the eigenvector-functions $\boldsymbol{\theta}_k(x)$ of the operator $\mathbf{F}^*$ corresponding to its eigennumbers $\sigma_k{}^\circ$, i.e.,

$$\mathbf{F}^*\boldsymbol{\theta}_k(x)=\sigma_k{}^\circ\boldsymbol{\theta}_k(x)$$

form a basis in $L_2{}^2([-1,1],V)$.

3°. If $\mathbf{q}(x,t)$ and $(\boldsymbol{\delta}(t)-\mathbf{g}(x,t))$ are continuous vector-functions in t in $L_2([-1,1],V)$ they can be represented in the form

$$\mathbf{q}(x,t)=v_1{}^i(t)\,\mathbf{p}_1{}^i(x)+\Sigma v_k(t)\,\boldsymbol{\theta}_k(x)$$
$$\boldsymbol{\delta}(t)-\mathbf{g}(x,t)=h_1{}^i(t)\,\mathbf{p}_1{}^i(x)+\Sigma h_k(t)\,\boldsymbol{\theta}_k(x)$$

where $v_1{}^i(t)$, $h_1{}^i(t)$, $v_k(t)$, $h_k(t)$ are continuous in t.

The relationships to find the eigennumbers and vector-functions have the form

$$\boldsymbol{\theta}_k(x)=\Sigma y_{p(k)}^l\mathbf{p}_p{}^l(x),\quad \Sigma r_{mn}^{ij}y_{n(k)}^j=\sigma_k{}^\circ y_{m(k)}^i$$

Noting that

$$\mathbf{k}_*{}^i(x)=\Sigma k_{*k}^i\boldsymbol{\theta}_k(x),\quad \boldsymbol{\alpha}(t)\,x=(^2/_3)^{1/2}\alpha^i(t)\,\mathbf{p}_1{}^i(x),\quad \mathbf{i}^i=\Sigma I_k{}^i\boldsymbol{\theta}_k(x)$$

by virtue of Theorem 4 we will have from (2.4) and (2.5)

$$v_k(t)=(\mathbf{I}^*+\mathbf{W}_k{}^*)\{[h_k(t)-(\mathbf{I}^*-\mathbf{L}_2{}^*)\,v_1{}^i(t)\,k_{*k}^i]/[c(t)+\sigma_k{}^\circ]\}$$
$$\alpha^i(t)=(^3/_2)^{1/2}[c(t)(\mathbf{I}^*-\mathbf{L}_1{}^*)\,v_1{}^i(t)+(\mathbf{I}^*-\mathbf{L}_2{}^*)(\Sigma k_{*k}^i v_k(t)+$$
$$E^{ij}v_1{}^j(t))-h_1{}^i(t)],\quad v_1{}^i(t)=(^3/_2)^{1/2}m^i(t)$$
$$p^i(t)=\Sigma I_k{}^i v_k(t)=2^{1/2}\Sigma v_k(t)\,y_{0(k)}^i$$
$$\mathbf{W}_k{}^*f(t)=\int_{\tau_r}^{t}f(\tau)\,R_k{}^3(t,\tau)\,d\tau,\quad t\in[\tau_r,\tau_{r+1}]$$

where $R_k{}^3(t,\tau)$ is the resolvent of the kernel $K_k{}^\circ(t,\tau,\sigma_k{}^\circ)$.

Theorem 5. The solution of Eqs.(2.4) under conditions (2.5) exists in the selected classes of functions (vector-functions), is unique, and can be found with previous assigned accuracy by the methods described.

Note that the series representing the solutions converge in the mean; if

$$\int_{-1}^{1}\mathbf{K}(x,\xi)\cdot\cdot\mathbf{K}(x,\xi)\,d\xi<Z=\text{const},\quad x\in[-1,1] \tag{5.1}$$

then they converge regularly; moreover, if the vector-function $\mathbf{g}(x,t)$ is continuous in x and the tensor-function $\mathbf{K}(x,\xi)$ is continuous in the large, i.e.,

$$\int_{-1}^{1}|\mathbf{K}(x_1,\xi)-\mathbf{K}(x_2,\xi)|\,d\xi<\varepsilon,\ |x_1-x_2|<\delta \tag{5.2}$$
$$|\mathbf{M}(x,\xi)|=(\mathbf{M}(x,\xi)\cdot\cdot\mathbf{M}(x,\xi))^{1/2}$$

then the solutions are continuous functions (vector-functions) in x (see /15/).

We also note that the spectrum of the operator $\mathbf{A}^*$ does not agree with the spectra of the operators $\mathbf{B}^*$, $\mathbf{C}^*$, $\mathbf{F}^*$. This enables us to investigate problem (2.4), (2.5) in the formulations of Sects.3-5 in the spectrum of $\mathbf{A}^*$.

6. We will now consider the system of Eqs.(1.9) with additional conditions of type (1.7). It is seen that they represent (2.4) and (2.5) written component by component. It can be shown that the operator $\mathbf{A}^*$ that occurs when solving contact problems is completely continuous, selfadjoint, and positive-definite from $L_2([-1,1],V)$ into $L_2([-1,1],V)$, and its kernel satisfies conditions (5.1) and (5.2). Therefore, the solutions of the fundamental systems of two-dimensional integral equations have been constructed.

The following modified formulations are possible in probems on systems of attached stamps.

1°. The settlings and angles of rotation are known, it is required to find the contact stresses, forces and moments (see Sect.2).

2°. The forces and moments are known, it is required to find the contact stresses, settlements, and angles (see Sect.3).

3°. The forces and angles of rotation are known, it is required to find the contact stresses, settlements and moments (see Sect.4).

4°. The settlements and moments are knonw, it is required to find the contact stresses, forces and angles of rotation (see Sect.5).

It is natural that at the times τ_k of attachment (removal) of the stamps, the desired quantities can have finite jumps in values since the system of resolving equations changes. The jumps of the desired quantities can appear even in the case of a system of stamps fixed in a given time interval if the functions being given undergo jumps in this interval.

The need to verify the physical content of the solution at each instant, i.e., the presence of compressive stresses under the stamps, should especially be noted. The time at which a change in the sign of the stress occurs at at least one point is the time the stamp begins to peel off from the base, and the proposed methods are not applicable for a further investigation of the process. The presence of compressive stresses under all the stamps at a certain time does not indicate that changes in the signs of the stresses could not have ocurred at previous times; consequently, it is important to investigate the whole time interval in which the process develops.

Considering the peeling off not to have occurred, we formulate a correspondence principle for one special case. Let the viscoelastic bases under consideration age homogeneously and let their layers be fabricated from one material at identical times, i.e., in (2.4)

$$c(t)=c^\circ,\quad \mathbf{L}_1^*=\mathbf{L}_2^*=\mathbf{L}^*,\quad (\mathbf{I}^*-\mathbf{L}^*)^{-1}=(\mathbf{I}^*+\mathbf{N}^*),\quad \mathbf{g}(x,t)=\mathbf{0}$$

then for systems of simultaneously applied stamps with flat bases the following assertions are valid.

1) In the formulation 1° $\boldsymbol{\delta}(t)=\boldsymbol{\delta}^\circ(t)$, $\boldsymbol{\alpha}(t)=\boldsymbol{\alpha}^\circ(t)$, $\mathbf{q}(x,t)=(\mathbf{I}^*+\mathbf{N}^*)\,\mathbf{q}^\circ(x,t)$, $\mathbf{p}(t)=(\mathbf{I}^*+\mathbf{N}^*)\,\mathbf{p}^\circ(t)$, $\mathbf{m}(t)=(\mathbf{I}^*+\mathbf{N}^*)\,\mathbf{m}^\circ(t)$.

2) In the formulation 2° $\mathbf{q}(x,t)=\mathbf{q}^\circ(x,t)$, $\mathbf{p}(t)=\mathbf{p}^\circ(t)$, $\mathbf{m}(t)=\mathbf{m}^\circ(t)$, $\boldsymbol{\delta}(t)=(\mathbf{I}^*-\mathbf{L}^*)\,\boldsymbol{\delta}^\circ(t)$, $\boldsymbol{\alpha}(t)=(\mathbf{I}^*-\mathbf{L}^*)\,\boldsymbol{\alpha}^\circ(t)$.

3) In the formulation 3° for given zero angles of rotation $\boldsymbol{\alpha}(t)=\boldsymbol{\alpha}^\circ(t)=\mathbf{0}$, $\mathbf{q}(x,t)=\mathbf{q}^\circ(x,t)$, $\mathbf{p}(t)=\mathbf{p}^\circ(t)$, $\mathbf{m}(t)=\mathbf{m}^\circ(t)$, $\boldsymbol{\delta}(t)=(\mathbf{I}^*-\mathbf{L}^*)\,\boldsymbol{\delta}^\circ(t)$.

Here the symbol ° denotes solutions of problems without taking account of creep (elastic problems).

Assertions 1) and 2) are a generalization of those known for isolated stamps while 3) yields a new result: in order for stamps of a certain system applied simultaneously to a viscoelastic homogeneous ageing base not to experience a mismatch, forces and moments obtained from the solution of the analogous elastic problem should be applied; the stresses here agree with the elastic stresses while the settlement will vary according to the law from 3). The formulation 4° yields no physically meaningful correspondence principle.

We will present some other useful formulas. In the formulated contact problems (see (1.5), (2.3), /5/)

$$k^{ij}(x,\xi)=\pi^{-1}\int_0^\infty \frac{L(u)}{u}\cos\left[\frac{u}{\lambda}(\zeta_i x+\eta_i-\zeta_j\xi-\eta_j)\right]du \tag{6.1}$$

by taking into account that $\{P_k^*(x)\}$ in (2.3) are Legendre polynomials and

$$f_{mn}^{ij}(u)=\sqrt{\frac{(2m+1)(2n+1)}{\zeta_i\zeta_j}}\,\lambda\,\frac{L(u)}{u^2}\,J_{1/2+m}\left(\frac{\zeta_i u}{\lambda}\right)J_{1/2+n}\left(\frac{\zeta_j u}{\lambda}\right)$$

we obtain by using /20/

$$R_{mn}^{ij}=\int_0^\infty f_{mn}^{ij}(u)\cos(\eta_i-\eta_j)\frac{u}{\lambda}\,du,\quad \rho_{mn}^{ij}=\int_0^\infty f_{mn}^{ij}(u)\sin(\eta_i-\eta_j)\frac{u}{\lambda}\,du$$

$$r_{mn}^{ij}=\begin{cases}(-1)^{\frac{m+n-l}{2}}R_{mn}^{ij} & (l=0,\ m \text{ and } n - \text{ even};\ l=2,\ m \text{ and } n - \text{ odd})\\ (-1)^{\frac{m+n-k}{2}}\rho_{mn}^{ij} & (k=1,\ m - \text{ even and } n - \text{ odd};\ k=-1 \text{ otherwise})\end{cases}$$

We note that a kindred problem about the attachment of viscoelastic covers to an elastic half-plane was studied in /21/, where the solution was constructed by the method of orthogonal polynomials with the investigation of an infinite system of Volterra integral equations.

Example. We consider a concrete foundation with constant elastic characteristics. We represent the creep measure in the form /8/

$$C(t, \tau) = (C_0 + A_0 e^{-\beta\tau})(1 - e^{-\gamma(t-\tau)})$$

and we are given the following values /8, 22/:

$$C_0 E = 0.5522,\ A_0 E = 4,\ \nu = 0,1$$
$$\tau_0 = 0,\ \beta = 0.031\ \text{day}^{-1},\ \gamma = 0.06\ \text{day}^{-1}$$

The inhomogeneous ageing parameter μ /10, 23, 24/ is the inhomogeneous ageing characteristic of the upper layer. We will consider the lower layer to be rigidly clamped, then /5/ (see (6.1))

$$L(u) = \frac{2\varkappa \operatorname{sh} 2u - 4u}{2\varkappa \operatorname{ch} 2u + 4u^2 + 1 + \varkappa^2}, \quad \varkappa = 3 - 4\nu$$

Furthermore, in conformity with (1.5), we take

$$c(t) = 0.2,\ \lambda = 24,\ \zeta_1 = 1,\ \zeta_2 = 2,\ \eta_1 = 0,\ \eta_2 = 5$$
$$M_1(t) = M_2(t) = 0,\ P_1(t) = 1,\ P_2(t) = 2,\ g_1(x) = g_2(x) = 0$$

i.e., the length of the contact line of the first stamp $(b_1 - a_1)$ is half the length of the contact line of the second $(b_2 - a_2)$. The spacing between them is $(b_1 - a_1)$. The force acting on the second stamp is four times greater than that for the first stamp, they are applied centrally (the moments and zero). The thicknesses of the layers are characterized by $c(t)$ and λ. The first stamp is applied at time 1 and the second at the time τ_2. The actual time of application of the first stamp τ_1 is measured in days and is a time scale coefficient (see (1.5)).

We will investigate the behaviour of the fundamental dimensionless characteristics (see (1.5)) under a homogeneous ($\mu = 1, \tau_1 = 20$ days and $\tau_1 = 100$ days), natural inhomogeneous (the age of the upper layer diminishes with height; $\mu = 10, \tau_1 = 100$ days) and artificial inhomogeneous (the age of the upper layer grows with height; $\mu = 0,1, \tau_1 = 20$ days) ageing processes as a function of the time of application of the second stamp τ_2. The curves on the graphs for the three ageing cases will be denoted by solid, dashed, and dash-dot lines, respectively. To identify the state under the different stamps, we construct the graphs in coordinates, where $x_0 = x^*\zeta_i + \eta_i$, i.e., for the first stamp $(i = 1)\ -1 \leqslant x_0 \leqslant 1$, and for the second stamp $(i = 2)\ 3 \leqslant x_0 \leqslant 7$ (we will henceforth omit the zero subscript on the x). The values of the contact pressures magnified ten times are plotted along the q axes. The dependences of the settlements on time are constructed in a real scale on the δ and t axes, and the angles of rotation magnified a hundredfold are on the α_1 and α_2 and t axes. The curves of the settlements and angles of rotation for the first and second stamps are denoted by dark and light circles, respectively.

The contact pressure distributions under the stamps are shown in Fig.1 for the cases of natural and homogeneous ageing at the time $t = 1.5$. For curves *1* and *2* the time of application of the second stamp is $\tau_2 = 1$ and $\tau_2 = 1.5$ respectively. We recall that the first stamp is always applied at time 1. In the homogeneous ageing case, the stress distributions under the first stamp differ slightly for different times of application of the second stamp and are represented by one curve.

On simultaneous application of the stamps in the homogeneous ageing case, the stresses are independent of the time (see the correspondence principle) and are represented by the solid curves *1*. The same curves describe the stresses at the time $t = 1$ for simultaneous application of the stamps in the case of natural inhomogeneous ageing, while at the time $t = 1.5$ the stress distribution are described by the dashed curves *1*. Therefore, a tendency is noted towards substantial smoothing of the state of stress with time because of the natural inhomogeneous ageing.

Non-simultaneous stamp application for homogeneous ageing manifests a tendency towards an increase in the stress distribution non-uniformity under the stamps with time, although the instantaneous distributions at the time of application of the second stamp are more uniform than under simultaneous action. The effects mentioned are slight for the first stamp.

For non-simultaneous action of stamps in the case of natural inhomogeneous ageing, superposition occurs of the tendencies to smoothing of the state of stress in time due to the kind of ageing and to an increase in its non-uniformity because of the difference in the application times. Thus, under the action of the second stamp at the time $\tau_2 = 1.25$ the stress distributions are smoothed out under both stamps; they are smoothed out under the first stamp for $\tau_2 = 1.5$ but are not under the second. The time of application of the second stamp considerably affects the instantaneous distributions and their change with time for both stamps (compare the dashed curves *1* and *2*).

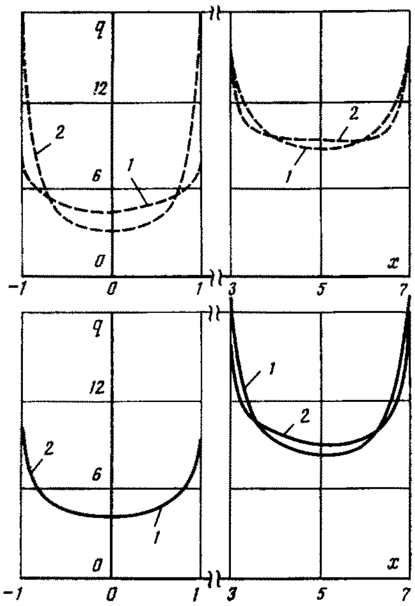

Fig.1

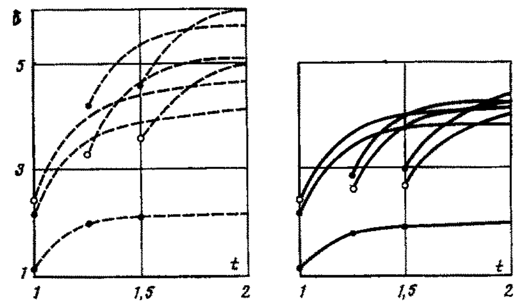

Fig.2

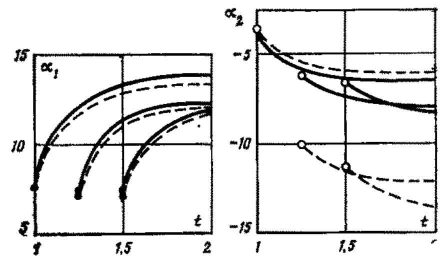

Fig.3

The contact stress distribution curves for the cases investigated have the shape of parabolas with maxima at the stamp edges remote from each other. The asymmetry of the distribution curves is not very clear, which is a result of the absence of moments (central application of the forces).

Fig.2 shows the dependences of the change in settlement on the time for the homogeneous and natural inhomogeneous ageing cases. The lower curves represents the change in settlement of the first stamp with time in the absence of the second stamp. The points mark the different times of application of the second stamp, where the settlement under the first stamp undergoes an unpward jump along the vertical between the dark points, while the open circle on this same vertical denotes the initial settlement of the second stamp.

Fig.3 shows the changes in the angles of stamp rotation with time as a function of the factors taken into account in studying settlement. The angle of rotation α_1 of the first stamp at the time of application of the second stamp undergoes a jump from zero to the value denoted by the dark point, while the initial value of the angle of rotation of the second stamp α_2 is denoted by the open circle. The kind of ageing has only a slight influence on the nature of the change in the angle under the first stamp but is substantial under the second.

Fig.4 shows the contact stress distributions under the stamps for the artificial inhomogeneous and homogeneous ageing cases at the time $t = 2$. For curves *1* and *2* we have $\tau_2 = 1$ and $\tau_2 = 2$, respectively. In the homogeneous ageing case one curve is shown for the first stamp for the reasons stipulated earlier.

The solid curves *1* show the stress distributions for simultaneous application of the stamps for the homogeneous case at any time and for the artificial ageing case at the time of action $t = 1$. The dash-dot curves *1* describe the stress distributions at the time $t = 2$ for simultaneous application of the stamps to an artificially inhomogeneously ageing base. The tendency towards a substantial increase of the stress distribution non-uniformity with time because of the artificial inhomogeneous ageing can be seen here.

The tendency to an increase in the contact stress distribution non-uniformity because of the non-simultaneity of stamp application was discussed earlier. Thus, for non-simultaneous stamp application in the case of an artificial inhomogeneous ageing, as a rule superposition occurs of these analogous tendencies, and the contact stress distribution non-uniformity increases under both stamps with time. However, the times of application of the second stamp were detected for the case noted when the stress distribution curves under the first stamp underwent qualitative changes, or peeling off of the first stamp occurred (peeling off of the upper from the lower layer). Indeed, the dash-dot curves *2* display the stress distributions at the time of application of the second stamp $\tau_2 = 2$. The contact pressures under the first stamp at this time are described by a two-humped curve with a minimum at the right edge (peeling off occurs at this layer for the time of application $\tau_2 = 2.2$). Note that a two-humped distribution curve is rectified with time and later acquires the usual parabolic shape.

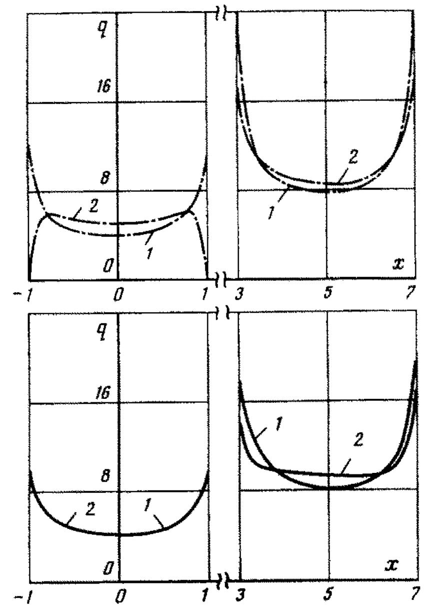

Fig.4

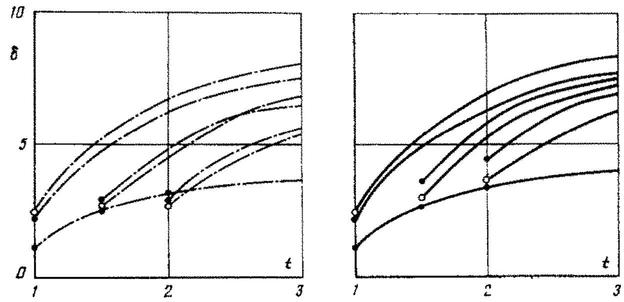

Fig.5

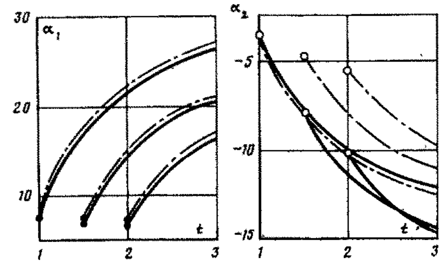

Fig.6

Fig.5 shows the dependences of the change in settlement under the stamps on time for the homogeneous and artificial inhomogeneous ageing cases. The notation is analogous to that in Fig.2. Here we merely note that the jump in the settlement under the first stamp is negative for artificial inhomogeneous ageing at $\tau_2 = t = 2$, i.e., it is somewhat elevated.

Fig.6 shows changes in the angles of rotation of the stamps with time matched to the cases investigated for the settlement. The solid curves on the graphs also enable one to track the influence of the time of homogeneous foundation fabrication on the stress, settlement and angles of rotation.

The author is grateful to N.Kh. Arutyunyan for discussing the research.

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Translated by M.D.F.

PMM U.S.S.R.,Vol.51,No.4,pp.535-537,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

PERIODIC SOLUTIONS OF SYSTEMS WITH GYROSCOPIC FORCES*

S.V. BOLOTIN

The lower limit for the number of periodic solutions of the equations of motion of a material point in n-dimensional Euclidean space under the effect of potential and gyroscopic forces is proved.

We consider a system with the gyroscopic forces /1/

$$(A(t)x^{\cdot})^{\cdot} = \Gamma x + U_x(x, t),\ x \in R^n \qquad (1)$$

where $A(t)$ is a symmetric positive-definite matrix, 2π-periodically continuously dependent on time, Γ is a constant skew-symmetric matrix of the gyroscopic forces, and the potential U depends 2π-periodically continuously on time, has continuous second derivatives with respect to the space variables and is periodic in them, for example

$$U(x+k, t) \equiv U(x, t) \qquad (2)$$

for all integer vectors $k \in Z \subset R^n$.

Theorem. If the system

$$(A(t)x^{\cdot})^{\cdot} = \Gamma x^{\cdot} \qquad (3)$$

has no non-constant 2π-periodic solutions, then system (1) has no less than $n+1$ different 2π-periodic solutions, and when multiplicity is taken into account, no less than 2^n. Solutions differing by a shift in the period of the potential are considered to be identical.

The conditions of the theorem mean that $A(t)x^{\cdot} = \Gamma x$ has no Floquet multipliers equal to one. If the potential U is small, then the assertion of the theorem can be obtained by methods of Poincaré perturbation theory.

System (1) is Lagrangian with the Lagrange function

$$L(x, x^{\cdot}, t) = {}^1/_2 (A(t)x^{\cdot}, x^{\cdot}) + {}^1/_2 (\Gamma x^{\cdot}, x) + U(x, t) \qquad (4)$$

We will seek 2π-periodic solutions of system (1) as critical points of the Hamilton action functional

*Prikl.Matem.Mekhan.,51,4,686-688,1987